Published by Institute of Physics Publishing for SISSA/ISAS
Received: *October 18, 2002*
Revised: *January 20, 2003*
Accepted: *January 28, 2003*

More on Penrose limits and non-local theories

Somdatta Bhattacharya and Shibaji Roy

Saha Institute of Nuclear Physics
1/AF Bidhannagar, Calcutta-700 064, India
E-mail: som@theory.saha.ernet.in, roy@theory.saha.ernet.in

Abstract: We obtain the Penrose limit of six dimensional Non-Commutative Open String ($NCOS_6$) theory and show that in the neighborhood of a particular null geodesic it leads to an exactly solvable string theory (unlike their counterparts in four or in other dimensions). We describe the phase structure of this theory and discuss the Penrose limit in different phases including Open D-string (OD1) theory. We compute the string spectrum and discuss their relations with the states of various theories at different phases. We also consider the case of general null geodesic for which the Penrose limit leads to string theory in the time dependent pp-wave background and comment on the renormalization group flow in the dual theory.

Keywords: p-branes, D-branes, Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

 http://jhep.sissa.it/archive/papers/jhep012003064/jhep012003064.pdf

Contents

1. Introduction

It is well-known that string theory in the pp-wave background can be obtained [1, 2] for type-IIB theory by taking the Penrose limit [3] on $AdS_5 \times S^5$ and has many interesting features. The most notable one is that the corresponding Green-Schwarz (GS) action in the light-cone gauge is exactly solvable [4]–[6]. So one can quantize the theory and obtain the string spectrum in a straightforward way. By the Maldacena conjecture [7]–[10] taking a Penrose limit (on the string/gravity side) amounts to going to a particular subsector of the $\mathcal{N} = 4$, SU(N) super Yang-Mills theory where both the conformal dimension Δ and the U(1) R-charge J of the gauge theory operators scales as Δ, $J \sim \sqrt{N}$ as $N \to \infty$, keeping g^2_{YM} and $\Delta - J$ fixed. Thus using the BMN [11] conjecture of exact correspondence between string theory in the pp-wave background and the subsector of gauge theory one can compute the anomalous dimensions of the gauge theory operators from the exact string spectrum and this has been generalized to many other AdS/CFT-like examples.

The consequence of taking Penrose limits and their implications in the subsector of dual nonconformal theories have been discussed in refs. [12]–[19]. In most cases Penrose limits on the gravity duals of these theories lead to string theories in time dependent pp-wave backgrounds. For a large class of such backgrounds, the equations of motion of the GS action can be solved exactly, but the quantization for such systems and the construction of states are still not well-understood [16, 13]. However, there is an intriguing connection between the associated time dependent quantum mechanical problem and the RG flow in the dual gauge theory [13]. On the other hand, it has been noticed that in six dimensions there exists a special null geodesic, for the gravity dual of both local [20] (ordinary YM or OYM) and non-local (Little String Theory[1] (LST) [19], noncommutative

[1]The existence of this theory has been argued in [21]–[23].

YM[2] (NCYM) [28] and open Dp-branes[3] (ODp) [33]) theories, in the neighborhood of which the Penrose limits lead to string theories in time independent pp-wave backgrounds. These are very similar to the maximally supersymmetric pp-wave limit of $AdS_5 \times S^5$. Since string theories in these cases are exactly solvable, it is straightforward to obtain the string spectrum and extract information about the states of the above mentioned theories in the subsector corresponding to the Penrose limit. Such discussions can be found for LST in [19], for 6-dimensional NCYM in [28] and for OD5 in [20].

In this paper we study the Penrose limit of the gravity dual of another class of non-local theories, namely, the noncommutative open string theory [34, 35] in six dimensions (NCOS$_6$) [36]. The supergravity configuration is given by the (F,D5) bound state [37] of type-IIB string theory in the so-called NCOS limit. We describe the phase structure of this theory and show how at various energies the theory is described by OYM$_6$, LST, NCOS$_6$ and OD1 theories. We obtain the Penrose limit of NCOS$_6$ theory in the neighborhood of a particular null geodesic by defining a scaling parameter in terms of the known parameters of the theory. We show that the Penrose limit in this case leads to an exactly solvable string theory in a time independent pp-wave background unlike the case of four or other dimensional NCOS theories. Here we find that the two of the eight bosonic coordinates of the string theory are massive and it contains both NSNS and RR three-form field strengths. We also discuss Penrose limits at different phases of this theory and as is known they all lead to solvable string theories. We study the quantization of the bosonic sector of the gravity dual of NCOS$_6$ theory in the Penrose limit, obtain the string spectrum and discuss their relations to the states in NCOS$_6$ theory in a particular subsector. Similar discussions are given for the different phases of this theory including the OD1 theory. Finally we obtain the Penrose limit for a more general null geodesic of both NCOS$_6$ theory and the OD1 theory. In this case we obtain string theories in time dependent pp-wave backgrounds which are not solvable. We briefly comment on the RG flow in the dual theory.

The organization of this paper is as follows. In section 2, we give the gravity dual description of NCOS$_6$ theory and describe its phase structure. The Penrose limit of this theory and various other theories at different phases for a particular null geodesic is discussed in section 3. The quantization of the bosonic sector and the string spectrum for the NCOS$_6$ and other theories at different phases are described in section 4. In section 5, we discuss the Penrose limits of NCOS$_6$ and OD1 theories for the general null geodesic and comment on the RG flow. Our conclusion is presented in section 6.

2. Supergravity description and the phase structure of NCOS$_6$

The gravity dual description of NCOS$_6$ can be obtained from the (F,D5) bound state configuration of type-IIB string theory in the so-called NCOS limit and is described in

[2] The gravity dual of 4-dimensional NCYM theory has been obtained in [24, 25] and in other dimensions in [26, 27].

[3] The existence of ODp theories have been shown in [29] (see also [30]) and their supergravity descriptions are given in [31, 32].

JHEP01(2003)064

refs. [36, 38]. The string metric, the dilaton and the other gauge fields have the forms,

$$
\begin{aligned}
ds^2 &= \epsilon(1+a^2r^2)^{1/2}ar\left[-(dx^0)^2+(dx^1)^2+\frac{1}{1+a^2r^2}\sum_{i=2}^{5}(dx^i)^2+\frac{MG_o^2\alpha'_{\rm eff}}{r^2}(dr^2+r^2d\Omega_3^2)\right]\\
e^{\phi} &= G_o^2(ar)\\
B &= \epsilon(ar)^2dx^0\wedge dx^1\\
F^{(3)} &= -2\epsilon M\alpha'_{\rm eff}\epsilon_3\\
A^{(4)} &= \frac{\epsilon^2}{G_o^2(1+a^2r^2)}dx^2\wedge dx^3\wedge dx^4\wedge dx^5\,.
\end{aligned} \tag{2.1}
$$

Here the parameter ϵ is defined as $\epsilon = \alpha'/\alpha'_{\rm eff}$, where $\sqrt{\alpha'}$ is the string length scale and $\sqrt{\alpha'_{\rm eff}}$ is the length scale of the NCOS$_6$ theory. The parameter $a^2 = \alpha'_{\rm eff}/(MG_o^2)$, where M is the number of D5-branes and G_o^2 is the coupling constant of NCOS$_6$ theory. 'r' is the energy parameter defined by $r = \tilde{r}/(\sqrt{\epsilon}\alpha'_{\rm eff})$, '$\tilde{r}$' being the radial coordinate transverse to the (F,D5) world-volume. $d\Omega_3^2$ is the line element of the unit 3-sphere transverse to (F,D5) world-volume and ϵ_3 is its volume-form. Note that in the NCOS limit $\alpha' \to 0$, with G_o^2 and $\alpha'_{\rm eff}$ fixed and so, ϵ is a small parameter.

The above supergravity description of NCOS$_6$ theory is valid as long as the curvature of the metric in (2.1) measured in α' unit as well as the dilaton remain small i.e.

$$
\alpha'\mathcal{R} = \frac{1}{MG_o^2ar(1+a^2r^2)^{1/2}} \ll 1 \tag{2.2}
$$

$$
e^{\phi} = G_o^2ar \ll 1\,. \tag{2.3}
$$

We will discuss the two cases (i) $ar \ll 1$ and (ii) $ar \gg 1$ separately. For case (i), the condition (2.2) implies $ar \gg 1/(MG_o^2)$ and so, along with the relation $ar \ll 1$, we have

$$
\frac{1}{MG_o^2} \ll ar \ll 1 \tag{2.4}
$$

where the curvature remains small. On the other hand the condition (2.3) implies $ar \ll 1/G_o^2$. So, if $G_o^2 \ll 1$, then we have the whole region (2.4) where both the curvature and the dilaton remain small and we have a valid supergravity description. However, we note that this conclusion is not quite correct. The reason is, in the region $ar \ll 1$ we expect the NCOS$_6$ theory to reduce to OYM$_6$ theory i.e. the supergravity configuration (2.1) should reduce to simple D5-brane configuration in the OYM-limit. It is easy to check that this would happen only if we set along with $ar \ll 1$ and $M \to \infty$, $G_o^2 \to \infty$, $\alpha'_{\rm eff} \to 0$, such that $G_o^2\alpha'_{\rm eff} = g_{\rm YM}^2/(2\pi)^3$. In this limit, (2.1) reduce to

$$
\begin{aligned}
ds^2 &= \alpha'\left[\frac{(2\pi)^{3/2}r}{\sqrt{M}g_{\rm YM}}\left(-(dx^0)^2+\sum_{i=1}^{5}(dx^i)^2\right)+\frac{g_{\rm YM}\sqrt{M}r}{(2\pi)^{3/2}}\left(\frac{dr^2}{r^2}+d\Omega_3^2\right)\right]\\
e^{\phi} &= \frac{g_{\rm YM}r}{(2\pi)^{3/2}\sqrt{M}}\\
F^{(3)} &= -2\alpha' M\epsilon_3
\end{aligned} \tag{2.5}
$$

with B and $A^{(4)}$ vanishing. This is exactly the D5-brane configuration in the OYM_6 limit [39]. So, when $ar \ll 1$, we take $G_o^2 \gg 1$. Therefore, the supergravity description remains valid not in the region (2.4) but

$$\frac{1}{MG_o^2} \ll ar \ll \frac{1}{G_o^2}\,. \tag{2.6}$$

Note that since we are in the OYM_6 region, a and G_o^2 do not have any obvious meaning. Substituting $a^2 = \alpha'_{\rm eff}/(MG_o^2)$ in (2.5) we find the range in this case as [39],

$$\frac{1}{\sqrt{M}} \ll r \ll \sqrt{M}\,. \tag{2.7}$$

When $ar \gg 1/G_o^2$, the dilaton becomes large and we have to go to the S-dual frame in order to have a valid supergravity description. In this case the D5-brane configuration will go over to the NS5-brane configuration by S-duality and OYM_6 limit will become the LST limit. Therefore, we will have the LST description in the region

$$\frac{1}{G_o^2} \ll ar \ll 1\,. \tag{2.8}$$

We will discuss how to obtain the LST supergravity description from OD1 theory later.

Now consider the case (ii) where $ar \gg 1$. Here we will assume $G_o^2 \ll 1$ and $\alpha'_{\rm eff}$ = fixed for the weakly coupled NCOS_6 theory. Note that the curvarture condition (2.2) in this case implies $ar \gg 1/(MG_o^2)^{1/2}$. Since for large M, $1/(MG_o^2)^{1/2} \ll 1$, the curvature condition is always satisfied. Also, the dilaton condition (2.3) implies $ar \ll 1/G_o^2$ and so, combining with $ar \gg 1$, we have the following range of ar for which NCOS_6 supergravity description remains valid,

$$1 \ll ar \ll \frac{1}{G_o^2}\,. \tag{2.9}$$

But when $ar \gg 1/G_o^2$, the dilaton becomes large and we have to go to the S-dual frame to have a valid supergravity description. Under S-duality the (F,D5) supergravity configuration becomes the (D1,NS5) supergravity configuration and the NCOS_6 limit goes over to the OD1 limit. To obtain the supergravity dual of OD1 theory we make an S-duality transformation in (2.1) which gives [38],

$$\begin{aligned}
ds^2 &= \epsilon G_{o(1)}^2 (1+a^2r^2)^{\frac{1}{2}} \left[-(dx^0)^2 + (dx^1)^2 + \frac{1}{1+a^2r^2}\sum_{i=2}^{5}(dx^i)^2 + \frac{N\tilde{\alpha}'_{\rm eff}}{r^2}(dr^2 + r^2 d\Omega_3^2)\right]\\
e^{\phi} &= \frac{G_{o(1)}^2}{ar}\\
dB &= 2\epsilon N G_{o(1)}^2 \tilde{\alpha}'_{\rm eff}\epsilon_3\\
A^{(2)} &= \epsilon (ar)^2 dx^0 \wedge dx^1\\
A^{(4)} &= \frac{\epsilon^2 G_{o(1)}^2}{1+a^2r^2} dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5\,.
\end{aligned} \tag{2.10}$$

Note that in writing (2.9) we have made use of the S-duality relations among the parameters of the two theories as [29, 38],

$$G_o^2 = \frac{1}{G_{o(1)}^2}\,, \qquad \text{and} \qquad G_o^2\alpha'_{\rm eff} = \tilde{\alpha}'_{\rm eff}\,, \tag{2.11}$$

where $G_{o(1)}^2$ is the coupling constant and $\sqrt{\tilde{\alpha}'_{\rm eff}}$ is the length scale in OD1 theory. Also, N is the number of NS5-branes and is equal to M. The parameter a^2 in NCOS$_6$ theory should be written in terms of the parameters of OD1 theory i.e.

$$a^2 = \frac{\alpha'_{\rm eff}}{MG_o^2} = \frac{G_{o(1)}^4\tilde{\alpha}'_{\rm eff}}{N}\,. \tag{2.12}$$

Note that when $ar \ll 1$, $G_{o(1)}^2 \to 0$ and $\tilde{\alpha}'_{\rm eff} = g_{\rm YM}^2/(2\pi)^3$ = fixed and this is precisely the LST limit. The supergravity configuration in this limit becomes [39]

$$\begin{aligned}
ds^2 &= -(dx^0)^2 + \sum_{i=1}^{5}(dx^i)^2 + \tilde{\alpha}'_{\rm eff}N\left(\frac{dr^2}{r^2} + d\Omega_3^2\right)\\
e^{\phi} &= \frac{(2\pi)^{3/2}\sqrt{N}}{g_{\rm YM}r}\\
dB &= 2N\tilde{\alpha}'_{\rm eff}\epsilon_3
\end{aligned} \tag{2.13}$$

with the other fields vanishing. This is exactly the supergravity configuration of LST. Note that in writing (2.12) we have multiplied by the $g_s^{-1} = 1/\epsilon G_{o(1)}^2$ of OD1 theory (with g_s, the string coupling constant), both the metric and dB in (2.9) so that both the S-dual metric of (NS5,D1) and the original metric of (F,D5) remain minkowskian [38].

So, to summarize, the phase structure of NCOS$_6$ theory is as follows. We have OYM$_6$ description in the range $1/(MG_o^2) \ll ar \ll 1/G_o^2$ and LST description in the range $1/G_o^2 \ll ar \ll 1$. On the other hand we have weakly coupled NCOS$_6$ description in the range $1 \ll ar \ll 1/G_o^2$ and we have weakly coupled OD1 description if $ar \gg 1/G_o^2$. In the first part when $ar \ll 1$, $G_o^2 \gg 1$ and for the second part when $ar \gg 1$, $G_o^2 \ll 1$.

3. Penrose limits

In this section we obtain the Penrose limit of NCOS$_6$ supergravity description given in (2.1) for a particular null geodesic and discuss the same for the other theories at different phases mentioned in the previous section. To obtain the Penrose limit of (2.1) we first scale the coordinates $x^{0,\ldots,5}$ as $x^{0,\ldots,5} \to \sqrt{MG_o^2\alpha'_{\rm eff}}x^{0,\ldots,5}$. Then by defining a new variable $ar = e^U$ we write (2.1) as,

$$\begin{aligned}
ds^2 &= R^2(1+e^{2U})^{\frac{1}{2}}e^U\left[-(dx^0)^2 + (dx^1)^2 + \frac{1}{1+e^{2U}}\sum_{i=2}^{5}(dx^i)^2 + dU^2 + d\Omega_3^2\right]\\
e^{\phi} &= G_o^2e^U\\
B &= R^2e^{2U}dx^0\wedge dx^1
\end{aligned}$$

$$
\begin{aligned}
F^{(3)} &= -2\frac{R^2}{G_o^2}\epsilon_3 \\
A^{(4)} &= \frac{R^4}{G_o^2(1+e^{2U})}dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \,.
\end{aligned} \tag{3.1}
$$

We write $d\Omega_3^2 = \cos^2\theta d\psi^2 + d\theta^2 + \sin^2\theta d\phi^2$ and look at a null geodesic for the metric in (3.1) restricted to (x^0, U, ψ)-plane. So, we set $x^{1,\ldots,5} = 0$, $\theta = 0$ and so, the effective lagrangian associated with this geodesic has the form,

$$
\mathcal{L} = -(1+e^{2U})^{1/2}e^U((x^0)')^2 + (1+e^{2U})^{1/2}e^U(U')^2 + (1+e^{2U})^{1/2}e^U(\psi')^2 \,, \tag{3.2}
$$

where 'prime' denotes the derivative with respect to the affine parameter along the geodesic. Since the lagrangian does not depend explicitly on x^0 and ψ, we get two constants of motion as,

$$
(1+e^{2U})^{1/2}e^U(x^0)' = E \,, \qquad (1+e^{2U})^{1/2}e^U\psi' = J \,. \tag{3.3}
$$

Substituting these in (3.2) and equating it to zero for the null geodesic we get the evolution equation for U as[4],

$$
(1+e^{2U})^{1/2}e^U U' = \sqrt{1-l^2} \,, \tag{3.4}
$$

where we have defined $l = J/E$ and scaled the affine parameter by E. Eq. (3.4) denotes the one parameter family of evolution equation for which $l^2 \leq 1$. We will discuss the general case $l < 1$ (we take $l \geq 0$) in section 5, and here we consider a special null geodesic for which $l = 1$. So, we have $U' = 0$ or $U =$ constant. Therefore the null geodesic is now restricted to (x^0, ψ)-plane as in the case of maximally supersymmetric $AdS_5 \times S^5$. The null geodesic is given by $U = U_0 =$ constant, $x^{1,\ldots,5} = \theta = 0$ and $x^0 = \psi = x^+$, where x^+ is the affine parameter. We now define a set of new coordinates as,

$$
\begin{aligned}
U &\to U_0 + (1+e^{2U_0})^{-1/4}e^{-U_0/2}x \\
\theta &\to (1+e^{2U_0})^{-1/4}e^{-U_0/2}z \\
x^1 &\to (1+e^{2U_0})^{-1/4}e^{-U_0/2}x^1 \\
x^{2,\ldots,5} &\to (1+e^{2U_0})^{1/4}e^{-U_0/2}x^{2,\ldots,5} \\
x^0 &\to x^+ + (1+e^{2U_0})^{-1/2}e^{-U_0}x^- \\
\psi &\to x^+ - (1+e^{2U_0})^{-1/2}e^{-U_0}x^- \,.
\end{aligned} \tag{3.5}
$$

By further rescaling the coordinates as $x^+ \to x^+$, $x^- \to x^-/R^2$, $x \to x/R$, $z \to z/R$, $x^{1,\ldots,5} \to x^{1,\ldots,5}/R$, $\phi \to \phi$ and taking $R \to \infty$[5], the supergravity configuration in (3.1)

[4]It should be mentioned here that the Euler-Lagrange equation for U yields a second order differential equation, since $\mathcal{L}$ explicitly depends on U. In general, the equation of motion for U is different from the null geodesic condition given below in (3.4). However, since the form of the metric in (3.2) has $-g_{00} = g_{\psi\psi} = g_{UU}$, it can be easily checked that the equation of motion for U is equivalent to the null geodesic codition as given below.

[5]In this limit, it is clear that we are in the neighborhood of the null geodesic just mentioned above.

takes the form,

$$\begin{aligned}
ds^2 &= -4dx^+dx^- - \vec{z}^2(dx^+)^2 + \sum_{i=1}^{5}(dx^i)^2 + dx^2 + d\vec{z}^2 \\
e^\phi &= G_o^2 e^{U_0} \\
H = dB &= 2\frac{e^{U_0}}{\sqrt{1+e^{2U_0}}}dx^+ \wedge dx^1 \wedge dx \\
F^{(3)} &= 2\frac{e^{-U_0}}{G_o^2\sqrt{1+e^{2U_0}}}dx^+ \wedge dz_1 \wedge dz_2 \\
F^{(5)} &= 0\,.
\end{aligned} \tag{3.6}$$

This is the Penrose limit of the NCOS$_6$ supergravity configuration. In the above $\vec{z}^2 = z_1^2 + z_2^2$ where $z_1 = z\cos\phi$ and $z_2 = z\sin\phi$. We note from the metric in (3.6) that only two of the eight bosonic coordinates, namely, $(z_1,\, z_2)$ have constant masses and the rest are massless. This will lead to an exactly solvable string theory. We can introduce arbitrary masses for z_1 and z_2 by scaling $x^\pm \to \mu^{\pm 1}x^\pm$ and then (3.6) become,

$$\begin{aligned}
ds^2 &= -4dx^+dx^- - \mu^2\vec{z}^2(dx^+)^2 + \sum_{i=1}^{5}(dx^i)^2 + dx^2 + d\vec{z}^2 \\
e^\phi &= G_o^2 e^{U_0} \\
H &= 2\mu\frac{e^{U_0}}{\sqrt{1+e^{2U_0}}}dx^+ \wedge dx^1 \wedge dx \\
F^{(3)} &= 2\mu\frac{e^{-U_0}}{G_o^2\sqrt{1+e^{2U_0}}}dx^+ \wedge dz_1 \wedge dz_2 \\
F^{(5)} &= 0\,.
\end{aligned} \tag{3.7}$$

We would like to mention that in taking the Penrose limit we have taken the scaling parameter $R^2 = \epsilon M G_o^2 \alpha'_{\rm eff} \to \infty$. This can be achieved by taking (*a*) $M \to \infty$, G_o^2, $\alpha'_{\rm eff}$ fixed or, (*b*) $M \to \infty$, $G_o^2 \to \infty$, $\alpha'_{\rm eff} \to 0$ such that $G_o^2\alpha'_{\rm eff} = g_{\rm YM}^2/(2\pi)^3 =$ fixed. For case (*a*) we are in NCOS$_6$ theory but for case (*b*) $a^2 = \alpha'_{\rm eff}/(MG_o^2) \to 0$ and so $ar \ll 1$. This is the region, as we have seen in the previous section, where we have OYM$_6$ supergravity description if $1/(MG_o^2) \ll ar \ll 1/G_o^2$ and LST supergravity description if $1/G_o^2 \ll ar \ll 1$. Note that for the OYM$_6$ supergravity description in the region mentioned, the metric and the dilaton remain the same as in (3.7) in the Penrose limit, but we have to replace $G_o^2 e^{U_0} = G_s^2 a r_0 = g_{\rm YM} r_0/((2\pi)^{3/2}\sqrt{M})$. But looking at the NSNS and RR 3-forms we find that $H = 2\mu(ar_0)dx^+ \wedge dx^1 \wedge dx$ vanishes, but, $F^{(3)} = (2\mu/(ar_0 G_o^2))dx^+ \wedge dz_1 \wedge dz_2$ does not vanish since $ar_0 G_o^2 \ll 1$. This is precisely the Penrose limit of the D5-brane (in the near horizon limit) discussed in [20]. For LST supergravity description in the region $1/G_o^2 \ll ar \ll 1$, we have to go to the S-dual frame and we will discuss how to obtain this from the OD1 theory later.

We have mentioned in the previous section that for $ar \gg 1$ and $G_o^2 \ll 1$, we have NCOS$_6$ description in the range $1 \ll ar \ll 1/G_o^2$ whose Penrose limit we have already discussed. But when $ar \gg 1/G_o^2$, the dilaton becomes large and we have to go to the S-dual description which is nothing but the gravity dual of OD1 theory given in (2.9). We now discuss the Penrose limit of this theory. In this case we scale the coordinates as, $x^{0,\ldots,5} \to \sqrt{N\tilde{\alpha}'_{\rm eff}}x^{0,\ldots,5}$, define a new coordinate $e^U = ar$ as before and define the scaling parameter as $R^2 = \epsilon N\tilde{\alpha}'_{\rm eff}G^2_{o(1)}$ (this is not the same as in NCOS$_6$ case), then (2.9) takes the form,

$$\begin{aligned}
ds^2 &= R^2(1+e^{2U})^{1/2}\left[-(dx^0)^2+(dx^1)^2+\frac{1}{1+e^{2U}}\sum_{i=2}^{5}(dx^i)^2+dU^2+d\Omega_3^2\right]\\
e^{\phi} &= \frac{G^2_{o(1)}}{e^U}\\
dB &= 2R^2\epsilon_3\\
A^{(2)} &= \frac{R^2}{G^2_{o(1)}}e^{2U}dx^0\wedge dx^1\\
A^{(4)} &= \frac{R^4}{G^2_{o(1)}(1+e^{2U})}dx^2\wedge dx^3\wedge dx^4\wedge dx^5\,. \qquad (3.8)
\end{aligned}$$

To obtain the Penrose limit we proceed exactly as in NCOS$_6$ case. The evolution equation in this case takes the form,

$$(1+e^{2U})^{1/2}U' = \sqrt{1-l^2}\,. \qquad (3.9)$$

The parameter 'l' was defined before and the the 'prime' denotes derivative with respect to the affine parameter along the geodesic in (x^0, U, ψ)-plane. Again we notice that for $l = 1$, $U' = 0$ or, $U =$ constant is a solution to (3.9). The null geodesic is given by $U = U_0 =$ constant, $x^{1,\ldots,5} = \theta = 0$ and $x^0 = \psi = x^+$ (the affine parameter) and is restricted to (x^0, ψ)-plane. The set of new coordinates we now define are as follows,

$$\begin{aligned}
U &\to U_0+(1+e^{2U_0})^{-1/4}x\\
\theta &\to (1+e^{2U_0})^{-1/4}z\\
x^1 &\to (1+e^{2U_0})^{-1/4}x^1\\
x^{2,\ldots,5} &\to (1+e^{2U_0})^{1/4}x^{2,\ldots,5}\\
x^0 &\to x^+ + (1+e^{2U_0})^{-1/2}x^-\\
\psi &\to x^+ - (1+e^{2U_0})^{-1/2}x^-\,. \qquad (3.10)
\end{aligned}$$

By further rescaling the coordinates as $x^+ \to \mu x^+$, $x^- \to x^-/(\mu R^2)$, $x \to x/R$, $z \to z/R$, $x^{1,\ldots,5} \to x^{1,\ldots,5}/R$, $\phi \to \phi$ and taking $R \to \infty$, the configuration (3.8) takes the following form in the new coordinates,

$$\begin{aligned}
ds^2 &= -4dx^+dx^- - \mu^2\vec{z}^2(dx^+)^2+\sum_{i=1}^{5}(dx^i)^2+dx^2+d\vec{z}^2\\
e^{\phi} &= G^2_{o(1)}e^{-U_0}\\
dB &= -\frac{2\mu}{\sqrt{1+e^{2U_0}}}dx^+\wedge dz_1\wedge dz_2
\end{aligned}$$

$$
\begin{aligned}
dA^{(2)} &= \frac{2\mu e^{2U_0}}{G^2_{o(1)}\sqrt{1+e^{2U_0}}} dx \wedge dx^+ \wedge dx^1 \\
F^{(5)} &= 0\,.
\end{aligned} \tag{3.11}
$$

This is the Penrose limit of the gravity dual of OD1 theory and has also been obtained in ref. [33]. In order to recover the Penrose limit of gravity dual of LST from here in the region $G^2_{o(1)} \ll ar \ll 1$, we also have to set $G^2_{o(1)} \to 0$ and $\tilde{\alpha}'_{\rm eff} = g^2_{\rm YM}/(2\pi)^3 =$ fixed. The metric would have the same form as given in (3.11). The dilaton $e^\phi = G^2_{o(1)}/(ar_0) = (2\pi)^{3/2}\sqrt{N}/(g_{\rm YM} r_0)$, $dB = -2\mu dx^+ \wedge dz_1 \wedge dz_2$ which remains finite, but $dA^{(2)} = (2\mu G^2_{o(1)}\tilde{\alpha}'_{\rm eff}/N) r_0^2 dx \wedge dx^+ \wedge dx^1$ vanishes. This is exactly the Penrose limit of the gravity dual of LST discussed in [19].

4. Quantization and the string spectrum

In this section we discuss the quantization of the bosonic sector of the closed string theory obtained in the previous section by taking the Penrose limit of the gravity dual of NCOS$_6$ theory as well as various other theories at different phases. We will obtain the string spectrum and discuss their relations to the states in various dual theories. The GS action for the bosonic part has the form,

$$
-4\pi\alpha' S_b = \int d^2\sigma \left[\eta^{ab} G_{\mu\nu}\partial_a x^\mu \partial_b x^\nu + \epsilon^{ab} B_{\mu\nu}\partial_a x^\mu \partial_b x^\nu\right], \tag{4.1}
$$

where $\eta^{ab} = {\rm diag}(-1,\,1)$ is the world-sheet metric and $\epsilon^{\tau\sigma} = 1$. We first rename the coordinates as $(z_1,\, z_2,\, x^5,\, x^4,\, x^3,\, x^2,\, x^1,\, x) \equiv (z_1,\, z_2,\, z_3,\, z_4,\, z_5,\, z_6,\, z_7,\, z_8)$ and then for the background (3.7) the above action takes the form in the light-cone gauge as,

$$
-4\pi\alpha' S_b = \int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \left[\eta^{ab}\partial_a z_i \partial_b z_i + \mu^2 z_k^2 + 4\mu \frac{e^{U_0}}{\sqrt{1+e^{2U_0}}} z_8 \partial_\sigma z_7\right], \tag{4.2}
$$

where $i = 1, \ldots, 8$ and $k = 1, 2$. We have also used the light-cone gauge $x^+ = \tau$. Let us define $Y = (z_7 + i z_8)/2$, then the equations of motion following from (4.2) are,

$$
\begin{aligned}
\eta^{ab}\partial_a\partial_b z_k - \mu^2 z_k &= 0\,, \qquad \text{for} \qquad k = 1,2 \\
\eta^{ab}\partial_a\partial_b z_l &= 0\,, \qquad \text{for} \qquad l = 3,\ldots,6 \\
\eta^{ab}\partial_a\partial_b Y + 2i\mu \frac{e^{U_0}}{\sqrt{1+e^{2U_0}}}\partial_\sigma Y &= 0 \\
\eta^{ab}\partial_a\partial_b \bar{Y} - 2i\mu \frac{e^{U_0}}{\sqrt{1+e^{2U_0}}}\partial_\sigma \bar{Y} &= 0\,.
\end{aligned} \tag{4.3}
$$

We solve these equations by Fourier expanding the various coordinates as,

$$
\begin{aligned}
z_i &= \sum_{n=0}^{\infty}\left[\frac{1}{\sqrt{4p^+\omega_n}}\alpha_n^i e^{-i\omega_n\tau + in\sigma/(\alpha' p^+)} + \frac{1}{\sqrt{4p^+\omega_{-n}}}(\alpha_n^i)^\dagger e^{i\omega_{-n}\tau - in\sigma/(\alpha' p^+)}\right] \\
Y &= \sum_{n=0}^{\infty}\left[\frac{1}{\sqrt{4p^+\omega_n}}\alpha_n^+ e^{-i\omega_n\tau + in\sigma/(\alpha' p^+)} + \frac{1}{\sqrt{4p^+\omega_{-n}}}(\alpha_n^-)^\dagger e^{i\omega_{-n}\tau - in\sigma/(\alpha' p^+)}\right]
\end{aligned} \tag{4.4}
$$

and similarly for $\bar{Y}$, where,

$$\begin{aligned}
\omega_n &= \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}}, \qquad \text{for} \quad z_1, z_2 \\
&= \frac{|n|}{\alpha' p^+}, \qquad \text{for} \quad z_3, \dots, z_6 \\
&= \sqrt{\frac{n^2}{(\alpha' p^+)^2} + 2\mu \frac{e^{U_0}}{(1+e^{2U_0})^{1/2}} \frac{n}{\alpha' p^+}}, \qquad \text{for} \quad Y, \bar{Y}
\end{aligned} \tag{4.5}$$

and we find that the oscillators obey the commutation relations,

$$\left[\alpha_m^i,\, (\alpha_n^j)^\dagger\right] = i\delta_{mn}\delta^{ij}, \quad \left[\alpha_m^+,\, (\alpha_n^+)^\dagger\right] = \left[\alpha_m^-,\, (\alpha_n^-)^\dagger\right] = i\delta_{mn}\,, \tag{4.6}$$

where $i, j = 1, \dots, 6$. So, the bosonic part of the light-cone hamiltonian takes the form,

$$\begin{aligned}
2p^- &= \sum_n \Bigg[N_n^{(k)} \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}} + N_n^{(l)} \frac{|n|}{\alpha' p^+} + \\
&\qquad + (N_n^+ + N_{-n}^-) \sqrt{\frac{n^2}{(\alpha' p^+)^2} + 2\mu \frac{e^{U_0}}{(1+e^{2U_0})^{1/2}} \frac{n}{\alpha' p^+}} \Bigg].
\end{aligned} \tag{4.7}$$

Now in order to relate the string spectrum to the states in NCOS$_6$ theory we write from (3.5)

$$\frac{\partial}{\partial x^+} = \frac{\partial}{\partial x^0} + \frac{\partial}{\partial \psi}, \qquad \frac{\partial}{\partial x^-} = \frac{e^{-U_0}(1+e^{2U_0})^{-1/2}}{R^2} \left(\frac{\partial}{\partial x^0} - \frac{\partial}{\partial \psi} \right). \tag{4.8}$$

In terms of the generators of the original x^0 (before rescaling by $\sqrt{MG_o^2 \alpha'_{\text{eff}}}$) we get,

$$\begin{aligned}
\frac{2p^-}{\mu} &= i\frac{\partial}{\partial x^+} = \sqrt{MG_o^2 \alpha'_{\text{eff}}} E - J_1 \\
2\mu p^+ &= i\frac{\partial}{\partial x^-} = \frac{e^{-U_0}(1+e^{2U_0})^{-1/2}}{R^2} \left(\sqrt{MG_o^2 \alpha'_{\text{eff}}} E + J_1 \right),
\end{aligned} \tag{4.9}$$

where we have used $i\partial/\partial x^0 = \sqrt{MG_o^2 \alpha'_{\text{eff}}} E$ and $-i\partial/\partial\psi = J_1$. We thus find a correspondence between the string spectrum and the states in the NCOS$_6$ theory with energy and U(1) R-charge,

$$\sqrt{MG_o^2 \alpha'_{\text{eff}}} E\,, \qquad J_1 \sim MG_o^2 \alpha'_{\text{eff}} \to \infty\,, \quad \text{with} \quad \sqrt{MG_o^2 \alpha'_{\text{eff}}} E - J_1 = \text{fixed}\,. \tag{4.10}$$

Thus the spectrum of strings in the background (3.7) is the same as the spectrum of NCOS$_6$ theory in the regime (4.10). For $R^2 \to \infty$, we have,

$$\mu p^+ R^2 = e^{-U_0}(1+e^{2U_0})^{-1/2} J_1 \qquad \Rightarrow \quad \alpha' p^+ = \frac{e^{-U_0}(1+e^{2U_0})^{-1/2} J_1}{\mu M G_o^2}. \tag{4.11}$$

So, the light-cone energy now takes the form,

$$\frac{2p^-}{\mu} = \sum_n \left[N_n^{(k)} \sqrt{1 + \frac{M^2 G_o^4 e^{2U_0}(1+e^{2U_0})n^2}{J_1^2}} + N_n^{(l)} \frac{|n| M G_o^2 e^{U_0}(1+e^{2U_0})^{1/2}}{J_1} + \right.$$
$$\left. + (N_n^+ + N_{-n}^-) \sqrt{\frac{M^2 G_o^4 e^{2U_0}(1+e^{2U_0})n^2}{J_1^2} + \frac{2 e^{2U_0} M G_o^2 n}{J_1}} \right]. \tag{4.12}$$

There are three terms in the light-cone energy expression in (4.12). The first term corresponds to the two massive bosons z_1, z_2, the second term corresponds to the four free massless bosons $z_3, \ldots, z_6$ and the third term corresponds to the two complex interacting bosons Y, $\bar{Y}$. The massive bosons when written in terms of two complex massive bosons $z = (z_1 + iz_2)/2$ and $\bar{z} = (z_1 - iz_2)/2$, will carry a $\mathrm{U}(1)_2$-charge corresponding to the angular coordinate ϕ, while the rest of the bosons are $\mathrm{U}(1)_2$-charge neutral. Actually the gravity dual of NCOS_6 theory before taking the Penrose limit has $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ isometry of S^3 and $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ is the subgroup of this group corresponding to the isometries of ψ and ϕ of $d\Omega_3^2$. The first one corresponds to the R-charge J_1 introduced earlier. Also, we note that the NCOS_6 theory does not have the full 6-dimensional Poincare invariance because of the presence of the electric field along z_7-direction which is proportional to z_8 (or vice-versa) as can be seen from (3.7). Thus we have another $\mathrm{U}(1)_3$-charge carried by the bosonic fields Y and $\bar{Y}$ and the other bosons are neutral under this charge. This is the reason we have a further splitting of the light-cone energy (the last term in (4.12)) for the bosons Y and $\bar{Y}$. We would like to point out that such a splitting did not happen for the case of 6-dimensional NCYM theory studied in [28] and the effect of magnetic field there was unobservable in the spectrum. The reason might be that we obtained the spectrum only for the closed string sector and the effect might be observable in the open string sector. However, in this case, we see the effect of electric field even in the closed string sector.

We have seen that if ar lies between $1/(MG_o^2) \ll ar \ll 1/G_o^2$, then the Penrose limit of NCOS_6 theory (3.7) reduces to that of OYM_6 theory and the metric has the same form as given in (3.7), the dilaton is given as $e^\phi = g_{\mathrm{YM}} r_0 / ((2\pi)^{3/2}\sqrt{M})$, while $dB = 0$. The RR 3-form is non-vanishing. So, the GS action for the bosonic sector will have the same form as given in (4.2) without the last term. The bosonic part of the light-cone hamiltonian will then be given as,

$$2p^- = \sum_n \left[N_n^{(k)} \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}} + N_n^{(l)} \frac{|n|}{\alpha' p^+} \right], \tag{4.13}$$

where $k = 1, 2$ correspond to the massive bosons z_1, z_2 and $l = 3, \ldots, 8$ correspond to the rest of the free massless bosons. To relate the string spectrum with the states of OYM_6 theory we first find

$$\frac{2p^-}{\mu} = i\frac{\partial}{\partial x^+} = i\frac{\partial}{\partial x^0} + i\frac{\partial}{\partial \psi} = \frac{\sqrt{M} g_{\mathrm{YM}}}{(2\pi)^{3/2}} E - J_1$$
$$2\mu p^+ = i\frac{\partial}{\partial x^-} = \frac{e^{-U_0}}{R^2}\left(i\frac{\partial}{\partial x^0} - i\frac{\partial}{\partial \psi}\right) = \frac{e^{-U_0}}{R^2}\left(\frac{\sqrt{M} g_{\mathrm{YM}}}{(2\pi)^{3/2}} E + J_1\right). \tag{4.14}$$

The states of OYM$_6$ theory have energy and $UU(1)$ R-charge

$$\frac{\sqrt{M}g_{\rm YM}}{(2\pi)^{3/2}}E\,, \qquad J_1\sim\frac{Mg_{\rm YM}^2}{(2\pi)^3}\to\infty\,, \quad \text{with} \quad \frac{\sqrt{M}g_{\rm YM}}{(2\pi)^{3/2}}E-J_1=\text{fixed}\,. \tag{4.15}$$

Taking $R^2\to\infty$ in (4.14), the light-cone energy takes the form,

$$\frac{2p^-}{\mu}=\sum_n\left[N_n^{(k)}\sqrt{1+\frac{Mg_{\rm YM}^2}{(2\pi)^3}\frac{e^{2U_0}n^2}{J_1^2}}+N_n^{(l)}\frac{|n|\sqrt{M}g_{\rm YM}}{(2\pi)^{3/2}J_1}e^{U_0}\right]. \tag{4.16}$$

Here $e^{U_0}=r_0$. A similar form of light-cone energy has also been obtained in [20].

We next discuss the spectrum for the OD1 theory in the Penrose limit by looking at the configuration given in (3.11). The light-cone GS action in this case takes the form,

$$-4\pi\alpha' S_b=\int d\tau\int_0^{2\pi\alpha' p^+}d\sigma\left[\eta^{ab}\partial_a z_i\partial_b z_i+\mu^2 z_k^2-\frac{2\mu}{\sqrt{1+e^{2U_0}}}z_2\partial_\sigma z_1\right], \tag{4.17}$$

where $i=1,\ldots,8$ and $k=1,2$. Defining $z=(z_1+iz_2)/2$, the equations of motion following from (4.17) have the forms,

$$\begin{aligned}
\eta^{ab}\partial_a\partial_b z_l &= 0\,, \qquad \text{for} \qquad l=3,\ldots,8\\
\eta^{ab}\partial_a\partial_b z-\mu^2 z+\frac{2i\mu}{\sqrt{1+e^{2U_0}}}\partial_\sigma z &= 0\\
\eta^{ab}\partial_a\partial_b \bar z-\mu^2 \bar z-\frac{2i\mu}{\sqrt{1+e^{2U_0}}}\partial_\sigma \bar z &= 0\,.
\end{aligned} \tag{4.18}$$

We can solve these equations by Fourier expanding the coordinates $z_{3,\ldots,8}$ as the first expression in (4.4) and z, $\bar z$ as the second expression in (4.4), where now ω_n will be of the forms,

$$\begin{aligned}
\omega_n &= \frac{|n|}{\alpha' p^+}\,, \qquad \text{for} \quad z_3,\ldots,z_8\\
&= \sqrt{\mu^2+\frac{n^2}{(\alpha' p^+)^2}+\frac{2\mu}{(1+e^{2U_0})^{1/2}}\frac{n}{\alpha' p^+}}\,, \qquad \text{for} \quad z,\bar z
\end{aligned} \tag{4.19}$$

and the oscillators satisfy the commutation relations,

$$\left[\alpha_m^i,\,(\alpha_n^j)^\dagger\right]=i\delta_{mn}\delta^{ij}\,, \qquad \left[\alpha_m^+,\,(\alpha_n^+)^\dagger\right]=\left[\alpha_m^-,\,(\alpha_n^-)^\dagger\right]=i\delta_{mn}\,, \tag{4.20}$$

where $i,j=3,\ldots,6$. So, the bosonic part of the light-cone hamiltonian would be given as,

$$2p^-=\sum_n\left[N_n^{(l)}\frac{|n|}{\alpha' p^+}+(N_n^++N_{-n}^-)\sqrt{\mu^2+\frac{n^2}{(\alpha' p^+)^2}+\frac{2\mu}{(1+e^{2U_0})^{1/2}}\frac{n}{\alpha' p^+}}\right]. \tag{4.21}$$

To relate the string spectrum with those of the OD1 theory we first find

$$\begin{aligned}
\frac{2p^-}{\mu} &= i\frac{\partial}{\partial x^+}=i\frac{\partial}{\partial x^0}+i\frac{\partial}{\partial\psi}=\sqrt{N\tilde\alpha'_{\rm eff}}E-J_1\\
2\mu p^+ &= i\frac{\partial}{\partial x^-}=\frac{(1+e^{2U_0})^{-1/2}}{R^2}\left(i\frac{\partial}{\partial x^0}-i\frac{\partial}{\partial\psi}\right)=\frac{(1+e^{2U_0})^{-1/2}}{R^2}\left(\sqrt{N\tilde\alpha'_{\rm eff}}E+J_1\right),
\end{aligned} \tag{4.22}$$

where we have used $i\partial/\partial x^0 = \sqrt{N\tilde{\alpha}'_{\rm eff}}E$ and $-i\partial/\partial\psi = J_1$. The corresponding states in OD1 theory will have energy and U(1) R-charge

$$\sqrt{N\tilde{\alpha}'_{\rm eff}}E \qquad \text{and} \qquad J_1 \sim N\tilde{\alpha}'_{\rm eff} \to \infty\,, \qquad \text{with} \quad \sqrt{N\tilde{\alpha}'_{\rm eff}}E - J_1 = \text{fixed}\,. \tag{4.23}$$

Thus the spectrum of strings in the background (3.11) is the same as those of the OD1 theory in the regime (4.21). For $R^2 \to \infty$, we find

$$\mu p^+ R^2 = (1+e^{2U_0})^{-1/2} J_1\,, \qquad \text{or} \qquad \alpha' p^+ = \frac{(1+e^{2U_0})^{-1/2} J_1}{\mu N}\,. \tag{4.24}$$

The light-cone energy therefore takes the form,

$$\frac{2p^-}{\mu} = \sum_n \left[N_n^{(l)} \frac{(1+e^{2U_0})^{1/2} N}{J_1} |n| + (N_n^+ + N_{-n}^-) \sqrt{1 + \frac{(1+e^{2U_0})N^2}{J_1^2} n^2 + \frac{2Nn}{J_1}} \right]. \tag{4.25}$$

We have seen that in the regime $G^2_{o(1)} \ll ar \ll 1$, the Penrose limit of the OD1 supergravity description reduces to that of the LST supergravity description. This is described after eq. (3.11). The spectrum of LST from the quantization of GS action has already been discussed in ref. [19] and we will not repeat it here.

5. Penrose limits for general null geodesics

In the previous sections we have discussed the Penrose limit of NCOS$_6$ supergravity description and various other theories at different phases for a particular null geodesic corresponding to the parameter $l = 1$. In this section we extend it for $l < 1$ and discuss Penrose limits of NCOS$_6$ and OD1 theories. The gravity dual of NCOS$_6$ theory is given in (3.1). By restricting the null geodesic in (x^0, U, ψ)-plane we obtained the evolution equation for U in (3.4) for the general value of the parameter l. Eq. (3.4) can be solved as,

$$\frac{1}{2} e^U \sqrt{1+e^{2U}} + \sinh^{-1} e^U = \sqrt{1-l^2}\, u\,, \tag{5.1}$$

where u is the affine parameter along the null geodesic. One can use this relation to formally express e^U as a function of u and let us call that function as g, i.e., $e^U = g(u)$ which satisfies (5.1). Now we make a coordinate change from (x^0, U, ψ) $\to$ (u, v, x) by the relations,

$$\begin{aligned} dU &= \frac{\sqrt{1-l^2}}{g\sqrt{1+g^2}} du \\ dx^0 &= \frac{1}{g\sqrt{1+g^2}} du + 2dv + l dx \\ d\psi &= \frac{l}{g\sqrt{1+g^2}} du + dx\,. \end{aligned} \tag{5.2}$$

By further rescaling the coordinates $u \to u$, $v \to v/R^2$, $\theta \to z/R$, $x \to x/R$, $x^{1,\dots,5} \to x^{1,\dots,5}/R$, $\phi \to \phi$ and taking $R \to \infty$, the metric in (3.1) reduces to

$$\begin{aligned} ds^2 &= -4dudv - \frac{l^2}{g\sqrt{1+g^2}}\vec{z}^2 du^2 + g\sqrt{1+g^2}(1-l^2)dx^2 + g\sqrt{1+g^2}d\vec{z}^2 + \\ &\quad + g\sqrt{1+g^2}(dx^1)^2 + \frac{g}{\sqrt{1+g^2}}\sum_{i=2}^{5}(dx^i)^2 \,. \end{aligned} \tag{5.3}$$

This is the form of the metric in 'Rosen' coordinates.[6] To write it in Brinkman form we define a new set of coordinates as,

$$\begin{aligned} u &\to x^+ \\ x^1 &\to \frac{1}{\sqrt{g}(1+g^2)^{1/4}}x^1 \\ x^{2,\dots,5} &\to \frac{(1+g^2)^{1/4}}{\sqrt{g}}x^{2,\dots,5} \\ \vec{z} &\to \frac{1}{\sqrt{g}(1+g^2)^{1/4}}\vec{z} \\ x &\to \frac{1}{\sqrt{1-l^2}\sqrt{g}(1+g^2)^{1/4}}x \\ v &\to x^- - \frac{1}{8}\left[\frac{(g\sqrt{1+g^2})'}{g\sqrt{1+g^2}}(x^2+\vec{z}^2+(x^1)^2) + \frac{(g/\sqrt{1+g^2})'}{(g/\sqrt{1+g^2})}\sum_{i=2}^{5}(x^i)^2\right]. \end{aligned} \tag{5.4}$$

Then the metric takes the form,

$$\begin{aligned} ds^2 &= -4dx^+dx^- - \left(m_z^2\vec{z}^2 + m_x^2(x^2+(x^1)^2) + m_{2,\dots,5}^2\sum_{i=2}^{5}(x^i)^2\right)(dx^+)^2 + \\ &\quad + (dx)^2 + d\vec{z}^2 + \sum_{i=1}^{5}(dx^i)^2 \,. \end{aligned} \tag{5.5}$$

Where the mass^2's associated with various coordinates are given as,

$$\begin{aligned} m_z^2 &= \frac{l^2}{g^2(1+g^2)} - \frac{[\sqrt{g}(1+g^2)^{1/4}]''}{\sqrt{g}(1+g^2)^{1/4}} \\ &= \frac{l^2}{e^{2U}(1+e^{2U})} + \frac{(1-l^2)(1+4e^{4U})}{4e^{2U}(1+e^{2U})^3} \\ m_x^2 &= -\frac{[\sqrt{g}(1+g^2)^{1/4}]''}{\sqrt{g}(1+g^2)^{1/4}} \\ &= \frac{(1-l^2)(1+4e^{4U})}{4e^{2U}(1+e^{2U})^3} \end{aligned}$$

[6]By ' ' we mean that strictly speaking the metric in Rosen coordinates should have $g_{uu} = 0$ (see for example, [40] and references therein) which is not the case here and also later in (5.9). However, that does not prevent us to obtain the desired Brinkman form of the metric in (5.5). The reason is, since u is merely renamed by x^+ to go to the Brinkman form, this term just shifts the mass^2 of the coordinates $\vec{z}$ as given in (5.6).

$$
\begin{aligned}
m^2_{2,\ldots,5} &= -\frac{[\sqrt{g}/(1+g^2)^{1/4}]''}{\sqrt{g}/(1+g^2)^{1/4}} \\
&= \frac{(1-l^2)}{4e^{2U}(1+e^{2U})^3}(1+8e^{2U})\,,
\end{aligned} \tag{5.6}
$$

where 'prime' in both (5.4) and (5.6) represents derivative with respect to the affine parameter $u = x^+$ along the geodesic. The second line in each of the mass^2 expressions in (5.6) are obtained by using the evolution equation (3.4). The important thing to note here is that all the mass^2 expressions are positive for $l < 1$ at all energies as opposed to some cases noted in the literature. However, since they are time dependent, it is not clear how to quantize and obtain the spectrum for the associated string theory. In this sense, for these general null geodesics the Penrose limit does not lead to a solvable string theory.

Now we discuss the Penrose limit of OD1 supergravity description for the general null geodesic. The supergravity configuration for OD1 theory is given in (3.8) and the evolution equation for U is given in (3.9). The solution of the equation has the form,

$$
\sqrt{1+e^{2U}} + \frac{1}{2}\ln\left[\frac{\sqrt{1+e^{2U}}-1}{\sqrt{1+e^{2U}}+1}\right] = \sqrt{1-l^2}\,u\,. \tag{5.7}
$$

Let us formally define $\sqrt{1+e^{2U}} = f(u)$ and make the following coordinate change,

$$
\begin{aligned}
dU &= \frac{\sqrt{1-l^2}}{f}du \\
dx^0 &= \frac{1}{f}du + 2dv + l\,dx \\
d\psi &= \frac{l}{f}du + dx\,.
\end{aligned} \tag{5.8}
$$

By further rescaling the coordinates $u \to u$, $v \to v/R^2$, $\theta \to z/R$, $x \to x/R$, $x^{1,\ldots,5} \to x^{1,\ldots,5}/R$, $\phi \to \phi$ and taking $R \to \infty$, the metric in (3.8) can be written in Rosen coordinates as,

$$
ds^2 = -4dudv - \frac{l^2}{f}\vec{z}^2du^2 + f(1-l^2)dx^2 + f d\vec{z}^2 + f(dx^1)^2 + \frac{1}{f}\sum_{i=2}^{5}(dx^i)^2\,. \tag{5.9}
$$

To write it in Brinkman form we define a new set of coordinates,

$$
\begin{aligned}
u &\to x^+ \\
x^1 &\to \frac{1}{\sqrt{f}}x^1 \\
x^{2,\ldots,5} &\to \sqrt{f}x^{2,\ldots,5} \\
\vec{z} &\to \frac{1}{\sqrt{f}}\vec{z} \\
x &\to \frac{1}{\sqrt{1-l^2}\sqrt{f}}x \\
v &\to x^- - \frac{1}{8}\left[\frac{f'}{f}(x^2+\vec{z}^2+(x^1)^2) + \frac{(f^{-1})'}{f^{-1}}\sum_{i=2}^{5}(x^i)^2\right].
\end{aligned} \tag{5.10}
$$

In these new coordinates the metric (5.9) takes exactly the same form as in the $NCOS_6$ theory given in (5.5), but the mass^2 expressions for the various coordinates are different and are given as follows,

$$\begin{aligned} m_z^2 &= \frac{l^2}{f^2} - \frac{(\sqrt{f})''}{\sqrt{f}} \\ &= \frac{l^2}{1+e^{2U}} + \frac{(1-l^2)e^{2U}}{4(1+e^{2U})^3}(e^{2U}-4) \\ m_x^2 &= -\frac{(\sqrt{f})''}{\sqrt{f}} \\ &= \frac{(1-l^2)e^{2U}}{4(1+e^{2U})^3}(e^{2U}-4) \\ m_{2,\ldots,5}^2 &= -\frac{(f^{-1/2})''}{f^{-1/2}} \\ &= \frac{(1-l^2)}{4(1+e^{2U})^3}e^{2U}(4-3e^{2U})\,. \end{aligned} \tag{5.11}$$

Here also 'prime' denotes derivative with respect to the affine parameter $u = x^+$. The second expressions of mass^2 in (5.11) are obtained by using the evolution equation (3.9). We note here that unlike in the $NCOS_6$ case the mass^2's are not always positive. In particular, we see from (5.11) that in the IR, m_x^2 becomes negative. However, we have seen in the previous sections that in the low energy the proper supergravity description should be that of $NCOS_6$ theory and not the OD1 theory where all the mass^2 are indeed positive. The presence of negative mass^2 indicates a quantum mechanical instability of the associated world-sheet theory and so OD1 theory in that case flows by RG to $NCOS_6$ theory where all the mass^2 of the world-sheet bosonic fields become positive. However, we would like to point out that in the UV where we know that OD1 is the proper supergravity description, the mass^2 are not always positive. For example, we note that $m_{2,\ldots,5}^2$ becomes negative in the UV. So, the appearance of negative mass^2 can not always be avoided by an RG flow argument. Some comments on the isuue of negative mass^2 has been made in ref. [13], but a better understanding of this is clearly needed.

6. Conclusion

In this paper we have studied the Penrose limit of the gravity dual of a class of non-local theories, namely, the NCOS theory in 6-dimensions. We discussed the phase structure of this theory and have shown how at various energies the theory is described by OYM_6, LST, $NCOS_6$ and OD1 theories. In particular, at low energies when $ar \ll 1$ and $G_o^2 \gg 1$, we get OYM_6 theory in the range $1/(MG_o^2) \ll ar \ll 1/G_o^2$ and LST in the range $1/G_o^2 \ll ar \ll 1$. On the other hand, at high energies when $ar \gg 1$, we have $NCOS_6$ theory in the range $1 \ll ar \ll 1/G_o^2$ and OD1 theory for $ar \gg 1/G_o^2$. We have obtained Penrose limits in the neighborhood of a special null geodesic and have shown that Penrose limits of the gravity duals of all these 6-dimensional theories lead to solvable string theories. We would like to emphasize that this is a specialty of 6-dimensional theories only. In fact, it is easy

JHEP01(2003)064

to see that Penrose limits of gravity duals of NCOS theories in other dimensions do not lead to solvable string theories. We have quantized the string theories obtained this way, constructed the light-cone hamiltonian and discussed their relations to the states of various theories at different phases of $NCOS_6$ theory. Finally, we also discussed Penrose limits of the gravity duals of both $NCOS_6$ and OD1 theories for the general null geodesic. In these cases Penrose limits yield string theories with time-dependent masses for the various bosonic fields corresponding to the target space coordinates and so they are not solvable (in the sense discussed in the paper). We have pointed out the appearance of negative mass^2 for OD1 theory and discussed the RG flow by which the mass^2 become positive in some cases.

References

[1] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of iib superstring theory*, *J. High Energy Phys.* **01** (2002) 047 [hep-th/0110242].

[2] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, *Class. and Quant. Grav.* **19** (2002) L87 [hep-th/0201081].

[3] R. Penrose, *Any space-time has a plane wave as a limit*, in *Differential geometry and relativity*, Reidel, Dordrecht 1976, pp. 271–275.

[4] R.R. Metsaev, *Type iib Green-Schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B 625** (2002) 70 [hep-th/0112044].

[5] R.R. Metsaev and A.A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, *Phys. Rev.* **D 65** (2002) 126004 [hep-th/0202109].

[6] J.G. Russo and A.A. Tseytlin, *On solvable models of type-IIB superstring in NS-NS and RR plane wave backgrounds*, *J. High Energy Phys.* **04** (2002) 021 [hep-th/0202179].

[7] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[8] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[9] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[10] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[11] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and PP waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[12] L.A. Pando Zayas and J. Sonnenschein, *On Penrose limits and gauge theories*, *J. High Energy Phys.* **05** (2002) 010 [hep-th/0202186].

[13] E.G. Gimon, L.A. Pando Zayas and J. Sonnenschein, *Penrose limits and RG flows*, hep-th/0206033.

[14] R. Corrado, N. Halmagyi, K.D. Kennaway and N.P. Warner, *Penrose limits of RG fixed points and PP-waves with background fluxes*, hep-th/0205314.

JHEP01(2003)064

[15] D. Brecher, C.V. Johnson, K.J. Lovis and R.C. Myers, *Penrose limits, deformed PP-waves and the string duals of $N = 1$ large-N gauge theory*, *J. High Energy Phys.* **10** (2002) 008 [hep-th/0206045].

[16] H. Fuji, K. Ito and Y. Sekino, *Penrose limit and string theories on various brane backgrounds*, *J. High Energy Phys.* **11** (2002) 005 [hep-th/0209004].

[17] J. Gomis and H. Ooguri, *Penrose limit of $N = 1$ gauge theories*, *Nucl. Phys.* **B 635** (2002) 106 [hep-th/0202157].

[18] E. Kiritsis and B. Pioline, *Strings in homogeneous gravitational waves and null holography*, *J. High Energy Phys.* **08** (2002) 048 [hep-th/0204004].

[19] V.E. Hubeny, M. Rangamani and E. Verlinde, *Penrose limits and non-local theories*, *J. High Energy Phys.* **10** (2002) 020 [hep-th/0205258].

[20] Y. Oz and T. Sakai, *Penrose limit and six-dimensional gauge theories*, *Phys. Lett.* **B 544** (2002) 321 [hep-th/0207223].

[21] M. Berkooz, M. Rozali and N. Seiberg, *On transverse fivebranes in M(atrix) theory on T^5*, *Phys. Lett.* **B 408** (1997) 105 [hep-th/9704089].

[22] N. Seiberg, *New theories in six dimensions and matrix description of M-theory on T^5 and $T^5/\mathbb{Z}_2$*, *Phys. Lett.* **B 408** (1997) 98 [hep-th/9705221].

[23] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, *Linear dilatons, NS5-branes and holography*, *J. High Energy Phys.* **10** (1998) 004 [hep-th/9808149].

[24] A. Hashimoto and N. Itzhaki, *Non-commutative Yang-Mills and the AdS/CFT correspondence*, *Phys. Lett.* **B 465** (1999) 142 [hep-th/9907166].

[25] J.M. Maldacena and J.G. Russo, *Large-N limit of non-commutative gauge theories*, *J. High Energy Phys.* **09** (1999) 025 [hep-th/9908134].

[26] M. Alishahiha, Y. Oz and M.M. Sheikh-Jabbari, *Supergravity and large-N noncommutative field theories*, *J. High Energy Phys.* **11** (1999) 007 [hep-th/9909215].

[27] J.X. Lu and S. Roy, *(p+1)-dimensional noncommutative Yang-Mills and D($p-2$) branes*, *Nucl. Phys.* **B 579** (2000) 229 [hep-th/9912165].

[28] S. Bhattacharya and S. Roy, *Penrose limit and NCYM theories in diverse dimensions*, hep-th/0209054.

[29] R. Gopakumar, S. Minwalla, N. Seiberg and A. Strominger, *OM theory in diverse dimensions*, *J. High Energy Phys.* **08** (2000) 008 [hep-th/0006062].

[30] T. Harmark, *Open branes in space-time non-commutative little string theory*, *Nucl. Phys.* **B 593** (2001) 76 [hep-th/0007147].

[31] M. Alishahiha, Y. Oz and J.G. Russo, *Supergravity and light-like non-commutativity*, *J. High Energy Phys.* **09** (2000) 002 [hep-th/0007215].

[32] I. Mitra and S. Roy, *(NS5,Dp) and (NS5,D($p+2$),Dp) bound states of type-IIB and type-IIA string theories*, *J. High Energy Phys.* **02** (2001) 026 [hep-th/0011236].

[33] M. Alishahiha and A. Kumar, *PP-waves from nonlocal theories*, *J. High Energy Phys.* **09** (2002) 031 [hep-th/0207257].

[34] N. Seiberg, L. Susskind and N. Toumbas, *Strings in background electric field, space/time noncommutativity and a new noncritical string theory*, *J. High Energy Phys.* **06** (2000) 021 [hep-th/0005040].

[35] R. Gopakumar, J.M. Maldacena, S. Minwalla and A. Strominger, *S-duality and noncommutative gauge theory*, *J. High Energy Phys.* **06** (2000) 036 [hep-th/0005048].

[36] T. Harmark, *Supergravity and space-time non-commutative open string theory*, *J. High Energy Phys.* **07** (2000) 043 [hep-th/0006023].

[37] J.X. Lu and S. Roy, *Non-threshold (F,Dp) bound states*, *Nucl. Phys.* **B 560** (1999) 181 [hep-th/9904129].

[38] R.G. Cai, I. Mitra, N. Ohta and S. Roy, *On decoupled theories in (5+1) dimensions from (F,D1,NS5,D5) supergravity configuration*, *J. High Energy Phys.* **12** (2001) 029 [hep-th/0111027].

[39] N. Itzhaki, J.M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large-N limit of theories with sixteen supercharges*, *Phys. Rev.* **D 58** (1998) 046004 [hep-th/9802042].

[40] M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos, *Penrose limits, supergravity and brane dynamics*, *Class. and Quant. Grav.* **19** (2002) 4753 [hep-th/0202111].